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On the polynomial solutions of the Calogero–Sutherland–Moser model

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Abstract. The symmetry of the polynomial solutions of the Calogero–Sutherland–Moser model which corresponds to the λ-deformed symmetric group S_N or the general linear group $GL(N)$ is treated using the relationship between S_N and the unitary group $SU(N)$. A λ-deformed relation S_N–$SU(N)$ is studied up to $N = 3$.

1. Introduction

Sogo [1] has solved exactly the eigenvalue problem for the Calogero–Sutherland–Moser [2, 3] (CSM) model which is an integrable one-dimensional Hamiltonian system. The homogeneous polynomial solutions of the equation

$$\left[\sum_{j=1}^{N} x_j \frac{\partial}{\partial x_j}\left(x_j \frac{\partial}{\partial x_j}\right) + \lambda \sum_{j<k} \frac{x_j + x_k}{x_j - x_k}\left(x_j \frac{\partial}{\partial x_j} - x_k \frac{\partial}{\partial x_k}\right)\right]\phi = \left(\frac{L}{2\pi}\right)^2 (E - E_0)\phi \qquad (1.1)$$

depend of λ (in the region $0 < \lambda \leqslant \frac{1}{2}$ and $1 \leqslant \lambda$) and Young diagrams play an essential role in classifying them. This fact occurs because the full symmetry of the CSM model is a λ-deformation of the symmetric group S_N or the general linear group $GL(N)$ (if we restrict ourselves to the unitary group the statement remains valid).

Besides, the eigenfunctions ϕ, specialized at $\lambda = 1$, coincide with the Schur functions multiplied by the dimensions of the S_N representations. The Schur functions are the characters of the group $SU(N)$ for the corresponding Young diagrams [4]. In our opinion that is the salient aspect of Sogo's article for it clarifies the importance of S_N and $SU(N)$ in the context of the CSM model.

Long ago, in a pioneering piece of work, Weyl [5] considered the relationship between the unitary group $SU(N)$ (or, in general, $GL(N)$) and the symmetric group S_N to deal with LS coupling ([5] pp 326–31 and 372–7 respectively).

Recently [6], the relationship between these groups has been treated using the induced characters of S_N and the general solution of LS coupling for a four-electron system has been obtained.

Pursuing this line of inquiry, our purpose in this paper is to generalize, to the CSM model symmetry, the results concerning the relation S_N–$SU(N)$, i.e. we shall formulate a λ-deformed relation between these groups, up to $N = 3$. Such a generalization gives a very specific content to the duality between S_N and $GL(N)$ mentioned in [1].

Finally, we must signal a slight change in notation respecting [6] in order to avoid confusion with the eigenfunctions ϕ, the induced character table of S_N will be denoted by F.

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2. The relation between S_N and $SU(N)$

In this section we shall recall some results concerning S_N and the relation between S_N and the unitary group $SU(N)$ [6].

Consider a partition $(N) = (N_1, \ldots, N_\ell)$ of N where $N_1 + N_2 + \cdots + N_\ell = N$, $N_1 \geqslant N_2 \geqslant \cdots \geqslant N_\ell > 0$. The Young diagrams corresponding to each partition will also be denoted by (N).

Let C be a class of S_N characterized by its cycle structure $(1^\alpha 2^\beta 3^\gamma \ldots)$. The permutations in C contain α 1-cycles, β 2-cycles etc, where $\alpha + 2\beta + 3\gamma + \cdots = N$.

Moreover, χ denotes the irreducible character table of S_N, F the induced character table of S_N, I the identity matrix and K a diagonal matrix whose elements are

$$[K_{jk}] = \delta_{jk}\frac{|C|}{N!} \qquad |C| \text{is the order of the } C \qquad |C| = \frac{N!}{1^\alpha \alpha! 2^\beta \beta! 3^\gamma \gamma! \ldots}$$

Δ is a lower triangular matrix such that Det $\Delta = 1$, $\forall N$. All these matrices are of dimensions $p(N) \times p(N)$. ($p(N)$ denotes the number of partitions of N.)

For S_N, the following equations hold:

$$\chi^{\mathrm{T}}\chi = K^{-1} \tag{2.1}$$

$$\chi K \chi^{\mathrm{T}} = I \tag{2.2}$$

$$F = \Delta\chi. \tag{2.3}$$

Note that from (2.1),

$$(\chi^{\mathrm{T}}\chi)^{-1} = (K^{-1})^{-1} \qquad \text{or} \qquad (\chi^{-1})(\chi^{-1})^{\mathrm{T}} = K. \tag{2.4}$$

From these expressions, some useful relations may be derived straightforwardly:

Proposition 1.

$$\Delta^{\mathrm{T}} = \chi K F^{\mathrm{T}}.$$

Proof. From (2.3)

$$\Delta = F\chi^{-1}$$

and

$$\Delta^{\mathrm{T}} = (\chi^{-1})^{\mathrm{T}} F^{\mathrm{T}}.$$

From (2.4), we get

$$(\chi^{-1})^{\mathrm{T}} = \chi K$$

therefore, the proposition follows.

Proposition 2.

$$\Delta\Delta^{\mathrm{T}} = FKF^{\mathrm{T}}.$$

Proof. From (2.3):

$$\chi^{-1} = F^{-1}\Delta \qquad \text{and} \qquad (\chi^{-1})^{\mathrm{T}} = \Delta^{\mathrm{T}}(F^{\mathrm{T}})^{-1}.$$

Using (2.4)

$$(\chi^{-1})(\chi^{-1})^{\mathrm{T}} = F^{-1}\Delta\Delta^{\mathrm{T}}(F^{\mathrm{T}})^{-1} = K$$

hence

$$FF^{-1}\Delta\Delta^{\mathrm{T}}(F^{\mathrm{T}})^{-1} = FK$$

or

$$\Delta\Delta^{\mathrm{T}}(F^{\mathrm{T}})^{-1} = FK.$$

Finally

$$\Delta\Delta^{\mathrm{T}} = FKF^{\mathrm{T}}.$$

All these results may be verified for $N = 3$; in this case we have:

$$\chi = \begin{array}{c} \\ (3) \\ (21) \\ (1^3) \end{array} \begin{array}{c} \begin{array}{ccc} (1^3) & (12) & (3) \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{pmatrix} \end{array} \qquad F = \begin{array}{c} \\ (3) \\ (21) \\ (1^3) \end{array} \begin{array}{c} \begin{array}{ccc} (1^3) & (12) & (3) \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 3 & 1 & 0 \\ 6 & 0 & 0 \end{pmatrix} \end{array}$$

$$K = \begin{pmatrix} \frac{1}{6} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} \qquad \text{and} \qquad \Delta = \begin{array}{c} \\ (3) \\ (21) \\ (1^3) \end{array} \begin{array}{c} \begin{array}{ccc} (3) & (21) & (1^3) \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \end{array}.$$

(For details see [6].)

Let us introduce the column matrix E of dim $p(N) \times 1$ of the Nth symmetric homogeneous polynomials $P_{(N)}$ specified by the Young diagram (N). $P_{(N)}$ is defined as

$$\sum x_1^{N_1} x_2^{N_2} x_3^{N_3} \ldots$$

where the sum is over all the partitions of N written in lexicographical order. Each row corresponds, respectively, to the partitions (N), $(N-1, 1), \ldots (1, 1, \ldots, 1)$.

For $N = 3$:

$$E = \begin{pmatrix} x_1^3 + x_2^3 + x_3^3 \\ x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2 \\ x_1x_2x_3 \end{pmatrix}.$$

Note that if instead of $GL(N)$, the unitary group $SU(N)$ is considered the variables of the symmetric functions must be complex numbers ε_j of unit absolute value.

$X(\varepsilon_1, \ldots, \varepsilon_N)$ are the characters of $SU(N)$ associated with the Young diagrams of the partitions of N.

We can summarize the relationship between $SU(N)$ and S_N by the equation

$$X = \chi KT \tag{2.5}$$

where T is the column matrix of the parameters $t_1, t_2, \ldots, t_{p(N)}$. Besides

$$T = F^{\mathrm{T}}E \tag{2.6}$$

From proposition 1, (2.5) and (2.6) the relation $SU(N)$–S_N may be written as

$$X = \Delta^{\mathrm{T}}E \tag{2.7}$$

(see [6] section IB). For $SU(3)$–S_3, equation (2.5) gives

$$X_{\square\square\square} = \frac{t_1}{6} + \frac{t_2}{2} + \frac{t_3}{3}$$

$$X_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}} = \frac{t_1}{3} - \frac{t_3}{3}$$

$$X_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} = \frac{t_1}{6} - \frac{t_2}{2} + \frac{t_3}{3}$$

(the reader may consult [5], p 375) and from equation (2.7) we get

$$
\begin{aligned}
X_{\square\square\square} &= (\varepsilon_1^3+\varepsilon_2^3+\varepsilon_3^3)+(\varepsilon_1^2\varepsilon_2+\varepsilon_1^2\varepsilon_3+\varepsilon_2^2\varepsilon_1+\varepsilon_2^2\varepsilon_3+\varepsilon_3^2\varepsilon_1+\varepsilon_3^2\varepsilon_2)+\varepsilon_1\varepsilon_2\varepsilon_3\\
X_{\substack{\square\square\\\square}} &= \qquad\qquad\qquad\quad (\varepsilon_1^2\varepsilon_2+\varepsilon_1^2\varepsilon_3+\varepsilon_2^2\varepsilon_1+\varepsilon_2^2\varepsilon_3+\varepsilon_3^2\varepsilon_1+\varepsilon_3^2\varepsilon_2)+2\varepsilon_1\varepsilon_2\varepsilon_3\\
X_{\substack{\square\\\square\\\square}} &= \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \varepsilon_1\varepsilon_2\varepsilon_3.
\end{aligned}
$$

It must be pointed out that when the relationship between the unitary and symmetric groups is applied to LS coupling only diagrams with no more than two columns are allowed by the Pauli exclusion principle [5, 6].

3. The generalization to the λ-deformed case

3.1. The λ-deformed symmetric group

In this section use will be made of two general results derived by Sogo [1]:

(a) A λ-deformed expression of the Frobenius formula is used to evaluate the λ-deformed character tables for S_N [1] appendix D). Hereafter we shall denote these tables by χ_λ.

(b) A procedure in which the eigenfunctions are expressed as linear combinations of the fully symmetric polynomials $P_{(N)}$ ([1], section II) to construct the eigenfunctions and eigenvalues of equation (1.1). We shall focus our attention on the coefficients of $P_{(N)}$ which are the elements of a lower triangular matrix Δ_λ, i.e. a λ-deformed matrix equivalent to Δ (see, for instance, equation (2.3)).

From (a) and (b) we get, for $N = 2$ and $N = 3$, λ-deformed expressions analogous to those displayed in section 2.:

$$\chi_\lambda K_\lambda \chi_\lambda^{\mathrm{T}} = I \tag{3.1}$$

$$\chi_\lambda^{\mathrm{T}} \chi_\lambda = K_\lambda^{-1} \tag{3.2}$$

$$F_\lambda = \Delta_\lambda \chi_\lambda \tag{3.3}$$

(Tables 1 and 2, respectively, illustrate the equations $K_\lambda = (\chi_\lambda^{-1})(\chi_\lambda^{-1})^{\mathrm{T}}$ and $\Delta_\lambda \Delta_\lambda^{\mathrm{T}} = F_\lambda K_\lambda F_\lambda^{\mathrm{T}}$ for $N = 3$.)

Moreover, taking into account the determinants of these matrices we get

$$\mathrm{Det}\,\chi_\lambda = \mathrm{Det}\,\chi \tag{3.4}$$

$$\mathrm{Det}\,K_\lambda = \mathrm{Det}\,K \tag{3.5}$$

$$\mathrm{Det}\,F_\lambda = \mathrm{Det}\,\Delta_\lambda \cdot \mathrm{Det}\,F. \tag{3.6}$$

Let us observe that

(i) through the application of (a) and (b) it is possible to obtain equations similar to (3.1)–(3.6) for $N > 3$ (an algorithm to evaluate the determinants of the character tables for finite groups may be found in [7] (p 65); and

(ii) equation (3.6) exhibits, in its right-hand side, $\mathrm{Det}\,\Delta_\lambda$ as a prefactor of $\mathrm{Det}\,F$. Hence, it may be said that $\mathrm{Det}\,F_\lambda$ is not invariant under λ-deformation.

Now we are going to examine in some detail the case $N = 3$. For the third degree the eigenfunctions are [1]

$$
\begin{aligned}
\phi_{(3)} &= 1\left\{P_{(3)} + \frac{3\lambda}{2+\lambda}P_{(2,1)} + \frac{6\lambda^2}{(2+\lambda)(1+\lambda)}P_{(1^3)}\right\}\\
\phi_{(2,1)} &= \frac{6}{2+\lambda}\left\{0 + P_{[2,1]} + \frac{6\lambda}{1+2\lambda}P_{(1^3)}\right\}\\
\phi_{(1^3)} &= \frac{6}{(1+\lambda)(1+2\lambda)}\{0+0+P_{(1^3)}\}
\end{aligned}
\tag{3.7}
$$

If $\lambda = 1$, we have

$$\begin{aligned} \phi_{(3)} &= 1\{P_{(3)} + P_{(2,1)} + P_{(1^3)}\} \\ \phi_{(2,1)} &= 2\{0 + P_{(2,1)} + 2P_{(1^3)}\} \\ \phi_{(1^3)} &= 1\{0 + 0 + P_{(1^3)}\}. \end{aligned}$$

So the eigenfunction ϕ specialized at $\lambda = 1$ coincides with the character of $GL(3)$—or $SU(3)$—multiplied by the degree of the irreducible representation of S_3. In general the prefactor of the λ-deformed character of $GL(N)$—or of $SU(N)$—is the λ-deformed degree of the S_N representations.

If we leave aside the prefactors, the right-hand side of equation (3.7) may be written as $\Delta^{\mathrm{T}} E$ (see equation (2.7)). Hence for the third degree, the λ-deformed Δ matrix is

$$\Delta_\lambda = \begin{pmatrix} 1 & 0 & 0 \\ \frac{3\lambda}{2+\lambda} & 1 & 0 \\ \frac{6\lambda^2}{(1+\lambda)(2+\lambda)} & \frac{6\lambda}{1+2\lambda} & \frac{6}{(1+\lambda)(1+2\lambda)} \end{pmatrix}. \tag{3.8}$$

For the third degree, this character table is [1]:

$$\chi_\lambda = \begin{array}{c} \\ (3) \\ (2,1) \\ (1^3) \end{array} \begin{array}{|ccc|} \multicolumn{1}{c}{(1^3)} & \multicolumn{1}{c}{(2,1)} & \multicolumn{1}{c}{(3)} \\ \hline 1 & 1 & 1 \\ \frac{6}{2+\lambda} & \frac{2-2\lambda}{2+\lambda} & \frac{-3\lambda}{2+\lambda} \\ \frac{6}{(1+\lambda)(1+2\lambda)} & \frac{-6\lambda}{(1+\lambda)(1+2\lambda)} & \frac{6\lambda^2}{(1+\lambda)(1+2\lambda)} \\ \hline \end{array} \tag{3.9}$$

Equation (3.3) allows us to evaluate the λ-deformed induced characters table, i.e., $F_\lambda = \Delta_\lambda \chi_\lambda$,

$$F_\lambda = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 1 & 0 \\ \frac{6(4\lambda^4 + 12\lambda^3 + 11\lambda^2 + 3\lambda + 6)}{(1+\lambda)^2(1+2\lambda)^2} & \frac{6\lambda(2\lambda^2 + 3\lambda - 5)}{(1+\lambda)^2(1+2\lambda)^2} & \frac{-6\lambda^2(2\lambda^2 + 3\lambda - 5)}{(1+\lambda)^2(1+2\lambda)^2} \end{pmatrix}$$

3.2. The λ-deformed relation S_N–$SU(N)$

To formulate the λ-deformed relationship between $SU(N)$ and S_N, it suffices to write equation (2.5) employing λ-deformed terms, i.e.

$$X_\lambda = \chi_\lambda K_\lambda T. \tag{3.10}$$

For $N = 3$, equation (3.9) and table 1 yield:

$$\begin{aligned} X_{\square\square\square} &= \frac{\lambda^2 t_1 + 3\lambda t_2 + 2t_3}{(\lambda+1)(\lambda+2)} \\ X_{\begin{smallmatrix}\square&\square\\ \square&\end{smallmatrix}} &= \frac{\lambda t_1 + (1-\lambda)t_2 - t_3}{2\lambda+1} \\ X_{\begin{smallmatrix}\square\\ \square\\ \square\end{smallmatrix}} &= \frac{t_1}{6} - \frac{t_2}{2} + \frac{t_3}{3} \end{aligned}$$

(in the appendix, the case $N = 2$ is considered).

For the time being, we ignore whether such a λ-deformed relation offers some new insight into the CSM model. Further studies are expected.

3.3. The relation between χ_λ and F_λ

The table χ of irreducible characters of S_N may be derived from the table F of induced characters [6]. In order to carry out this derivation, each row F_i of F must be considered as a vector which ought to be orthonormalized using the Gram–Schmidt method. In such a way the rows χ_i of χ are obtained. It has been shown that

$$\chi_i = F_i - \sum_{k=1}^{i-1}(F_i K \chi_k)\chi_k \qquad \text{for } i = 1, \chi_1 = F_1 \tag{3.11}$$

where χ_i and F_i are the ith rows of χ and F respectively. K is the diagonal matrix defined in section 2. Expression (3.11) may be written as

$$F_i = \chi_i + \sum_{k=1}^{i-1}(F_i K \chi_k)\chi_k.$$

Note that the coefficients of χ_k are the elements of the lower triangular matrix Δ appearing in equation (2.3). Besides, Det $\Delta = 1$, $\forall N$.

We may extend this procedure to the λ-deformed case in a direct manner. However, a crucial feature of the Gram–Schmidt procedure, i.e. Det $\Delta = 1$, does not hold if $\lambda \neq 1$: in general Det $\Delta_\lambda \neq 1$. Hence, this fact will modify the result.

For the λ-deformed case, expression (3.11) can be written as

$$\chi_{\lambda_i} = F_{\lambda_i} - \sum_{k=1}^{i-1}(F_{\lambda_i} K_\lambda \chi_{\lambda_k})\chi_{\lambda_k}. \tag{3.12}$$

As before, let us consider, as an illustration, $N = 3$. Our starting point is (3.12) and use must be made of K_λ (table 1). In principle we must expect, as a result, equation (3.9) with modifications due to the property: Det $\Delta_\lambda \neq 1$.

We have:

$$\chi_{\lambda_1} = F_{\lambda_1} = (1 \quad 1 \quad 1) \tag{3.13a}$$

$$\chi_{\lambda_2} = F_{\lambda_2} - (F_{\lambda_2} K_\lambda \chi_{\lambda_1})\chi_{\lambda_1} \qquad \text{or} \qquad \chi_{\lambda_2} = \left(\frac{6}{\lambda+2}\ \frac{2(1-\lambda)}{\lambda+2}\ \frac{-3\lambda}{\lambda+2}\right) \tag{3.13b}$$

$$\chi_{\lambda_3} = F_{\lambda_3} - (F_{\lambda_3} K_\lambda \chi_{\lambda_1})\chi_{\lambda_1} - (F_{\lambda_3} K_\lambda \chi_{\lambda_2})\chi_{\lambda_2} \tag{3.13c}$$

that is

$$\chi_{\lambda_3} = \left(\frac{36}{(\lambda+1)^2(2\lambda+1)^2}\ \frac{-36\lambda}{(\lambda+1)^2(2\lambda+1)^2}\ \frac{36\lambda^2}{(\lambda+1)^2(2\lambda+1)^2}\right)$$

i.e.

$$\chi_{\lambda_3} = \frac{6}{(\lambda+1)(2\lambda+1)}\left(\frac{6}{(\lambda+1)(2\lambda+1)}\ \frac{-6\lambda}{(\lambda+1)(2\lambda+1)}\ \frac{6\lambda^2}{(\lambda+1)(2\lambda+1)}\right).$$

From equation (3.8), we get

$$\det \Delta_\lambda = \frac{6}{(1+\lambda)(1+2\lambda)}$$

hence (3.13c) is nothing other than Det $\Delta_\lambda \cdot \chi_{\lambda_3}$. So, for $N = 3$, the first two rows of χ_λ obtained via the Gram–Schmidt method from F_λ correspond exactly with the first two rows of χ_λ. But the third, and last one, has, as a prefactor, the determinant of Δ_λ (see equation (3.6)).

Note. (Remark that if we do not play the game fairly, i.e. if instead of (3.8), we consider

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix}$$

throughout the calculations, χ_λ is derived from F_λ via the Gram–Schmidt method. This is a kind of *a contrario* verification of the assertions made so far about this point.)

Acknowledgments

The author is indebted to Pablo Velasquez for checking the results of section 3 as well as tables 1 and 2.

Appendix. λ-deformed S_2–$SU(2)$ relationship

For λ-deformed S_2, we have

$$\chi_\lambda = \begin{pmatrix} 1 & 1 \\ \dfrac{2}{1+\lambda} & \dfrac{-2\lambda}{1+\lambda} \end{pmatrix} \qquad \text{and} \qquad K_\lambda = \begin{pmatrix} \dfrac{5\lambda^2+2\lambda+1}{4(\lambda+1)^2} & \dfrac{2\lambda-\lambda^2-1}{4(\lambda+1)^2} \\ \dfrac{2\lambda-\lambda^2-1}{4(\lambda+1)^2} & \dfrac{5+2\lambda+\lambda^2}{4(\lambda+1)^2} \end{pmatrix}$$

hence

$$X_\lambda = \chi_\lambda K_\lambda T = \begin{pmatrix} \dfrac{\lambda}{\lambda+1} & \dfrac{1}{\lambda+1} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$$

i.e.

$$X_{\square\square} = \frac{\lambda}{\lambda+1} t_1 + \frac{1}{\lambda+1} t_2 \qquad X_{\boxminus} = \tfrac{1}{2} t_1 - \tfrac{1}{2} t_2$$

for $\lambda = 1$, the usual result is reproduced.

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Table 1. The λ-deformed K-matrix (third degree) (the ordinary result is reproduced if $\lambda = 1$).

$$K_\lambda = \begin{pmatrix} \frac{23}{18} - \frac{100\lambda^5 + 329\lambda^4 + 482\lambda^3 + 369\lambda^2 + 140\lambda + 20}{4(\lambda+1)^2(\lambda+2)^2(2\lambda+1)^2} & \frac{56\lambda^5 + 97\lambda^4 + 122\lambda^3 + 109\lambda^2 + 44\lambda + 4}{4(\lambda+1)^2(\lambda+2)^2(2\lambda+1)^2} - \frac{1}{3} & \frac{1}{18} - \frac{\lambda(\lambda^4 - 2\lambda^3 + 5\lambda^2 + 10\lambda + 4)}{(\lambda+1)^2(\lambda+2)^2(2\lambda+1)^2} \\ \frac{56\lambda^5 + 97\lambda^4 + 122\lambda^3 + 109\lambda^2 + 44\lambda + 4}{4(\lambda+1)^2(\lambda+2)^2(2\lambda+1)^2} - \frac{1}{3} & \frac{1}{2} - \frac{3(4\lambda^5 - 25\lambda^4 - 2\lambda^3 + 23\lambda^2 + 4\lambda - 4)}{4(\lambda+1)^2(\lambda+2)^2(2\lambda+1)^2} & \frac{\lambda^5 + 5\lambda^4 + 31\lambda^3 + 23\lambda^2 - 2\lambda - 4}{(\lambda+1)^2(\lambda+2)^2(2\lambda+1)^2} - \frac{1}{6} \\ \frac{1}{18} - \frac{\lambda(\lambda^4 - 2\lambda^3 + 5\lambda^2 + 10\lambda + 4)}{(\lambda+1)^2(\lambda+2)^2(2\lambda+1)^2} & \frac{\lambda^5 + 5\lambda^4 + 31\lambda^3 + 23\lambda^2 - 2\lambda - 4}{(\lambda+1)^2(\lambda+2)^2(2\lambda+1)^2} - \frac{1}{6} & \frac{\lambda^4 + 6\lambda^3 + 29\lambda^2 + 28\lambda + 8}{(\lambda+1)^2(\lambda+2)^2(2\lambda+1)^2} + \frac{1}{9} \end{pmatrix}$$

Table 2. λ-deformed proposition 2: $\Delta\Delta^{\mathrm{T}} = FKF^{\mathrm{T}}$ (third degree) (the ordinary result is reproduced if $\lambda = 1$).

$$\Delta_\lambda \Delta_\lambda^{\mathrm{T}} = F_\lambda K_\lambda F_\lambda^{\mathrm{T}} = \begin{pmatrix} 1 & \frac{3\lambda}{\lambda+2} & \frac{6\lambda^2}{(\lambda+1)(\lambda+2)} \\ \frac{3\lambda}{\lambda+2} & 10 - \frac{36(\lambda+1)}{(\lambda+2)^2} & \frac{6\lambda(7\lambda^3 + 8\lambda^2 + 8\lambda + 4)}{(\lambda+1)(\lambda+2)^2(2\lambda+1)} \\ \frac{6\lambda^2}{(\lambda+1)(\lambda+2)} & \frac{6\lambda(7\lambda^3 + 8\lambda^2 + 8\lambda + 4)}{(\lambda+1)(\lambda+2)^2(2\lambda+1)} & \frac{36(5\lambda^6 + 10\lambda^5 + 14\lambda^4 + 12\lambda^3 + 5\lambda^2 + 4\lambda + 4)}{(\lambda+1)^2(\lambda+2)^2(2\lambda+1)^2} \end{pmatrix}.$$